Midterm 2 - Review - Problems

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1 Determinants

Problem 1

Find $\det(A)$, where:

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 0 & 0 & 3 \\ 3 & 1 & 4 & 7 \\ 1 & 1 & 0 & 3 \end{bmatrix}$$

Problem 2

Find $\det(A)$, where:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix}$$

2 Diagonalization

Problem 3

Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, where:

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

3 Vector Spaces

Problem 4

Is the following set a subspace of $\mathbb{R}^3$?

$$V = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid a + b + c = 1 \right\}$$

Problem 5

Is the following set a subspace of $\mathbb{R}^3$? If yes, find a basis for W and $\dim(W)$:

$$W = Span \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Problem 6

If

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for $Row(A)$ and $Col(A)$

(b) Find $Rank(A)$ and $\text{Dim}Nul(A)$

Problem 7

Define $T : P_3 \to P_4$ by:

$$T(p) = \int_0^t p(x)dx$$

(Basically, $T(p)$ is the antiderivative of p without the constant)

(a) Show T is a linear transformation

(b) Find the matrix relative to the basis $\mathcal{B} = \{1, t, t^2, t^3\}$ of P_3 and $\mathcal{C} = \{1, t, t^2, t^3, t^4\}$ of P_4

Problem 8

If:

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \end{bmatrix} \right\}, \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Find $\mathcal{C} \overset{P}{\leftarrow} \mathcal{B}$ and use this to calculate $[\mathbf{x}]_{\mathcal{C}}$, where $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

4 Orthogonality

Problem 9

Find the orthogonal projection of $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}$ on $W = Span \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}$

Problem 10

Use the Gram-Schmidt process to find an orthonormal basis for:

$$W = Span \left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \right\}$$

Problem 11

Find a least-squares solution and the corresponding least-squares error of $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

5 True/False Extravaganza!

Problem 12

(a) Every square matrix is diagonalizable

(b) If Q is orthogonal, then $QQ^T = I$

(c) The columns of the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ are the $\mathcal{B}$-coordinate vectors of the vectors in $\mathcal{C}$

(d) The union of two subspaces of V is a subspace of V

(e) The intersection of two subspaces of V is a subspace of V

(f) If $A^2 = A$, then the only eigenvalues of A are 0 and 1

(g) If A is not invertible, then A is not diagonalizable.

(h) If V is a $4-$ dimensional vector space, then any set of 3 vectors cannot span V.

(i) If A is similar to B, then $\det(A) = \det(B)$

(j) If A is diagonalizable, then $det(A)$ is the product of the eigenvalues of A (counting multiplicities)

(k) For all $\mathbf{y}$ and subspaces W, $\mathbf{y} - \text{Proj}_W\mathbf{y}$ is orthogonal to W.